

CERN-TH.7159/94  
TPI-MINN-94/2-T  
UMN-TH-1235-94  
UND-HEP-94-BIG02  
[hep-ph/9402225]  
February 1994

# Heavy Quark Distribution Function in QCD and the AC<sup>2</sup>M<sup>2</sup> Model

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## Abstract

We show that the phenomenological AC<sup>2</sup>M<sup>2</sup> ansatz is consistent with QCD through order  $1/m_b$  in the description of  $B \rightarrow l\bar{\nu}_l + X_u$  and  $B \rightarrow \gamma + X_s$  transitions, including their energy spectra and differential distributions. This suggests a concrete realization for the QCD distribution function, which we call the “Roman” function. On the other hand the AC<sup>2</sup>M<sup>2</sup> model description of the end-point domain in  $B \rightarrow l\bar{\nu}_l + X_c$  is incompatible with QCD: a different distribution function enters the description of  $b \rightarrow c$  decays as compared to the transitions to the massless quarks. Both observations – the validity of the AC<sup>2</sup>M<sup>2</sup> -like description for heavy-to-light transitions and the emergence of the new distribution function in the  $b \rightarrow c$  case – are in contradiction to a recent claim in the literature. The intrinsic limitation of the AC<sup>2</sup>M<sup>2</sup> model could reveal itself in different values of the effective  $b$  quark mass from fits of the  $\Lambda_b$  and  $B$  decays.

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<sup>1</sup>During the academic year 1993/94

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1. In recent publications [1, 2] we have shown how the lepton and the photon energy spectra in semileptonic and radiative decays, respectively, of beauty hadrons can be predicted from QCD proper without recourse to phenomenological *ad hoc* assumptions; this is achieved by employing a heavy quark expansion in inverse powers of the beauty quark mass to deal with non-perturbative corrections. It turned out that the shape of the spectrum in the end-point region requires a special treatment (see also Refs. [3, 4]). The physical phenomenon behind that is quite transparent: the heavy quark  $Q$  moves inside the decaying hadron  $H_Q$  and this motion smears the lepton spectrum, most significantly in the end-point region. It has been noted [1, 2] that a rigorous QCD description – though having a similar origin – does not generically coincide with simple non-relativistic models of the ‘Fermi’ motion where the potential energy of the light degrees of freedom is always small as compared to their space-like momentum; the most popular model of this type is due to AC<sup>2</sup>M<sup>2</sup> [5]. Original generalities of the QCD approach are given in Refs. [6, 7, 8].

The AC<sup>2</sup>M<sup>2</sup> model describes the beauty meson  $B$  in very simple terms. It treats it as consisting of the heavy  $b$  quark plus a spectator with fixed mass  $m_{sp}$ ; the latter is of the order of a typical hadronic scale and represents a fit parameter. The spectator quark has a momentum distribution  $\Phi(|\vec{p}|)$  where  $\vec{p}$  is its three-dimensional momentum. The  $b$  quark then cannot possess a fixed mass. Indeed, the energy-momentum conservation implies for such a simple boundstate that the  $b$  quark energy  $E_b$  is given by  $M_B - \sqrt{|\vec{p}|^2 + m_{sp}^2}$ . The mass of the  $b$  quark equals  $E_b$  through order  $p$ ; thus one obtains a “floating”  $b$  mass

$$m_b^f \simeq M_B - \sqrt{|\vec{p}|^2 + m_{sp}^2} \quad (1)$$

which depends on  $|\vec{p}|$ . The lepton spectrum is first obtained from  $b$  quark decay in a moving frame where its momentum is  $-\vec{p}$ ; the quark mass is assumed to be  $m_b^f$ . Then the distribution is averaged with the weight function  $\Phi(|\vec{p}|)$  (further details can be found in the original work [5]). The Galilean transformation above introduces a Doppler smearing of the spectrum that is of order  $1/m_b$ ; the transverse Doppler shift arises only at order  $1/m_b^2$  and is ignored here.

The AC<sup>2</sup>M<sup>2</sup> model is extensively used in the analysis of the lepton energy spectrum in semi-leptonic decays. Surprising though it is, given all the naïveté of the model, it reproduces very well numerically the *shape* of the semi-leptonic spectra in its regular part, as it is derived from QCD (see [1]). In this paper we analyze in more detail what features of QCD are reflected in this model for the end-point spectrum. We will show that – contrary to a first impression – this model is built sufficiently well to reproduce correctly the pattern of the QCD description of the heavy quark motion provided one considers the transition of the heavy to a *massless* quark. Moreover, the specific choice adopted in AC<sup>2</sup>M<sup>2</sup> for the weight function  $\Phi(|\vec{p}|)$  [5] corresponds to a particular form of the genuine distribution function  $F(x)$  introduced in Ref. [2] which schematically can be viewed as

$$\Phi(|\vec{p}|) = \frac{4}{p_F^3 \sqrt{\pi}} e^{-\frac{\vec{p}^2}{p_F^2}} \longleftrightarrow \Phi(p_+) = \theta(p_+) \frac{1}{p_F \sqrt{\pi}} e^{-\frac{(p_+ - \frac{m_{sp}}{p_+})^2}{4p_F^2}} \quad (2)$$

where  $p_+$  is the light cone combination  $p_0 + p_z$  entering the QCD description (the exact relation to  $F(x)$  will be stated below). This represents an acceptable realization of the QCD distribution function meeting all necessary requirements (we will call it the Roman function <sup>1</sup>); it can easily accommodate improvements should they become necessary. At the same time the AC<sup>2</sup>M<sup>2</sup> model *per se* does not reproduce correctly the dependence on the final state quark mass  $m_q$  if the mass is large enough. As follows from the QCD analysis [2] the boundary lies at  $m_q \sim (\bar{\Lambda}m_b)^{1/2}$  and  $c$  quark is sufficiently heavy in this sense. Moreover, in the  $b \rightarrow c$  transitions the AC<sup>2</sup>M<sup>2</sup> model in its original form would yield spectra beyond the physical kinematic boundary. A good fit of the experimental lepton energy spectrum in  $b \rightarrow c$  is achieved [1] at a price of introducing an upper cut off in the heavy quark momentum depending on the final quark mass.

Recently a comparison of the QCD result and of the AC<sup>2</sup>M<sup>2</sup> model has been given in Refs. [9, 10], where it was pointed out that the lepton energy spectrum obtained from the AC<sup>2</sup>M<sup>2</sup> ansatz can be considered to be consistent with the QCD prediction in the sense that there are no corrections to order  $1/m_b$ . In this note we give a more detailed analysis. We also point out that the consideration of Refs. [3, 4] in the part concerning the AC<sup>2</sup>M<sup>2</sup> model is incorrect.

2. Some of the dynamical features in the spectra of charged leptons get obscured by the integration over the neutrino energy. They are seen in a cleaner way in the spectrum of photons in the radiative  $B \rightarrow \gamma + X_s$  transitions. This spectrum in the free quark approximation is just a monochromatic line:

$$\frac{d\Gamma^{(0)}}{dE} = \frac{\lambda^2}{4\pi} m_b^3 \delta(E - \frac{m_b}{2}) \quad (3)$$

where  $E$  is the photon energy,  $m_b$  is the (current)  $b$  quark mass, the strange quark is considered as massless and  $\lambda$  is the coupling constant determining the  $bs\gamma$  vertex as it emerges from the electroweak theory:

$$\mathcal{L}|_{b \rightarrow s\gamma} = \frac{i}{2} \lambda F_{\mu\nu} \bar{s}(1 + \gamma_5) \sigma_{\mu\nu} b. \quad (4)$$

In both the QCD approach [1, 2, 4] and in the AC<sup>2</sup>M<sup>2</sup> model [5] the monochromatic line of eq. (3) is transformed into a peak of a finite width due to the heavy quark motion. The width of the peak is of the order of a typical hadronic scale. Note, that in the AC<sup>2</sup>M<sup>2</sup> model the variation of the  $b$  quark mass (see below) also contributes to the width.

Gluon radiation will also lead to a smearing out of the photon line. Since our primary object here is to discuss non-perturbative effects like the Fermi motion, we will ignore perturbative gluon emission throughout most of this paper.

Non-perturbative corrections are incorporated in the AC<sup>2</sup>M<sup>2</sup> model through the three fit parameters  $m_{sp}$ ,  $m_q$  and  $p_F$ . Let us recall now some general expressions

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<sup>1</sup>The possible shape of the function shown in Figs. 1,2 demonstrates that it is definitely not of a sans serif type.

derived previously [2] from QCD for  $m_q = 0$  case. Neglecting the gluon radiative corrections one can write the spectrum for the inclusive radiative decay in the form

$$\frac{d\Gamma}{dE} = \frac{\lambda^2}{4\pi} m_b^3 \frac{2}{\bar{\Lambda}} F(x) \quad (5)$$

where  $\bar{\Lambda} = M_B - m_b$ ,

$$x = \frac{1}{\bar{\Lambda}}(2E - m_b) \quad , \quad (6)$$

and  $F(x)$  is the QCD distribution function. The moments of  $F(x)$  ,

$$a_n = \int dx x^n F(x) \quad , \quad n = 0, 1, \dots \quad (7)$$

are related to the expectation values of local operators taken between  $B$  mesons:

$$\frac{1}{2M_B} \langle B | \mathcal{S} \bar{b} \pi_{\mu_1} \dots \pi_{\mu_n} b - \text{traces} | B \rangle = a_n \bar{\Lambda}^n (v_{\mu_1} \dots v_{\mu_n} - \text{traces}). \quad (8)$$

Here  $\mathcal{S}$  is the symmetrization symbol and  $v_\mu$  is the four-velocity of the  $B$  meson. Moreover,

$$\pi_\mu = iD_\mu - m_b v_\mu \quad (9)$$

where  $D_\mu$  is the covariant derivative. Note that  $a_0 = 1$  and  $a_1 = 0$  (these relations hold up to  $1/m_b^2$  corrections which we disregard throughout). An important consequence of quantum chromodynamics is that the total width

$$\Gamma = \frac{\lambda^2}{4\pi} m_b^3 + \dots \quad (10)$$

receives no corrections at the level  $1/m_b$ ; they arise first at order  $1/m_b^2$  [7, 8].

It should be noted that the true distribution function  $F(x)$  is always one-dimensional. For in the QCD description different components of the momentum represented by  $\pi_\mu$  do not commute and the distribution function can be introduced only for a single combination of the components of the momentum. In the case at hand, when the final quark is massless, i.e.  $m_q = 0$ , it is the ‘light cone’ distribution function that enters (i.e. we actually deal with the distribution in  $\pi_0 + \pi_z$  where the z-axis is chosen along the direction of the photon momentum; for more details see Ref. [2]).

At first sight the two approaches – that of Refs. [1, 2, 4] on the one hand, and the AC<sup>2</sup>M<sup>2</sup> model [5] on the other – have completely different properties. Two main distinctions are obvious.

The AC<sup>2</sup>M<sup>2</sup> model introduces the distribution over the *three-dimensional* nonrelativistic momentum,  $\Phi(p)$  where  $p = |\vec{p}|$  is the absolute value of the  $b$  momentum. This distribution results in the Doppler smearing of the photon spectrum with the spread *linear* in  $1/m_b$ . Moreover, the AC<sup>2</sup>M<sup>2</sup> prescription replaces  $m_b$  by  $m_b^f$ , the floating quark mass, see eq. (1):

$$\frac{d\Gamma_{ACM}}{dE} = \frac{\lambda^2}{8\pi} \int p^2 dp dz \Phi(p) (m_b^f)^3 \delta(E - \frac{m_b^f}{2} - \frac{pz}{2}) \quad , \quad (11)$$

where  $z = \cos \theta$  and  $\theta$  is the angle between  $\vec{p}$  and the direction of the photon momentum. (Eq. (11) is obtained by substituting  $m_b$  by  $m_b^f$ , making the Lorentz boost and then convoluting the spectrum resulting from the parton approximation with  $\Phi(p)$ .)

At the same time, the QCD expansion operates in terms of the fixed, current mass of the the  $b$  quark,  $m_b$ . The effect of the Fermi motion enters via the light cone distribution function defined in eqs. (7), (8) (we remind the reader that the final  $s$  quark is treated as massless like the  $q$  quark before).

3. We will show now that in spite of these differences the two approaches can actually lead to the same photon spectrum. The  $AC^2M^2$  ansatz yields for the photon spectrum after averaging over the direction of  $\vec{p}$

$$\frac{d\Gamma_{ACM}}{dE} = \frac{\lambda^2}{8\pi} \theta(\epsilon) \int_{p_0^2(\epsilon)}^{\infty} dp^2 \Phi(p^2) \left( M_B - \sqrt{m_{sp}^2 + p^2} \right)^3, \quad (12)$$

with

$$\epsilon = M_B - 2E, \quad p_0^2(\epsilon) = \frac{1}{4} \left( \frac{m_{sp}^2}{\epsilon} - \epsilon \right)^2. \quad (13)$$

The spectrum is automatically cut off at the true kinematic boundary, i.e. at  $E = M_B/2$  by the step function for  $\epsilon < 0$ , irrespective of the particular choice of  $\Phi(p)$ . This is not surprising, of course, because the adoption of the floating  $b$  quark mass was dictated just by this requirement.

The shape of the spectrum is obtained by direct integration and depends on the specific form of  $\Phi(p)$ . The distribution  $\Phi(p)$  suggested in Ref. [5] and routinely used in experimental analyses is

$$\Phi(p) = \frac{4}{p_F^3 \sqrt{\pi}} e^{-\frac{p^2}{p_F^2}} \quad (14)$$

with the following normalization

$$\int_0^{\infty} dp p^2 \Phi(p) = 1.$$

Notice that the upper limit of integration (which must be  $M_B$ ) is extended to  $\infty$ . This introduces only an exponentially small error; below we will always use this extension of the upper limit of integration.

Using the ansatz of eq. (14) and keeping terms through order  $p_F/M_B$  one gets:

$$\frac{d\Gamma_{ACM}}{dE} = \left( \frac{\lambda^2 M_B^3}{4\pi} \right) \theta(\epsilon) \frac{2}{p_F \sqrt{\pi}} e^{-\frac{p_0^2(\epsilon)}{p_F^2}} \left[ 1 - \frac{3p_F}{M_B} e^{(\rho + \frac{p_0^2(\epsilon)}{p_F^2})} \Gamma\left(\frac{3}{2}, \rho + \frac{p_0^2(\epsilon)}{p_F^2}\right) + \mathcal{O}\left(\frac{p_F^2}{M_B^2}\right) \right] \quad (15)$$

where the energy  $E$  enters via  $\epsilon$ ,  $p_0^2(\epsilon)$  defined in eq. (13) and

$$\rho = \frac{m_{sp}^2}{p_F^2}. \quad (16)$$

The factor in the square brackets can be put to unity in discussing the *shape* of the spectrum in this approximation. The term proportional to  $p_F/M_B$  includes the

incomplete gamma function  $\Gamma(\alpha, x)$  and will be taken into account below to obtain the total width to  $1/m_b$  accuracy. The expression in eq. (15) exhibits a pronounced peak whose shape depends, of course, on the value of  $\rho$  (see Fig. 1). For  $\rho < 1$  it is rather asymmetric, a second gratifying feature of the  $AC^2M^2$  model, since it is in qualitative accord with the findings in QCD. As we will see later on, this model can provide an approximate description of the real world only when  $\rho$  is rather small.

Straightforward integration of the spectrum eq. (12) leads to

$$\Gamma_{ACM} = \frac{\lambda^2 M_B^3}{4\pi} \left[ 1 - \frac{3p_F\rho}{M_B\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right) + \mathcal{O}\left(\frac{p_F^2}{M_B^2}\right) \right] \quad (17)$$

where  $K_1$  is the McDonald function. Once  $\Gamma_{ACM}$  is expressed in terms of the quantity

$$m_b^{ACM} = M_B - \frac{p_F\rho}{\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right) \quad (18)$$

(which, as we will see in a moment, is nothing but the value of the floating mass  $m_b^f$  of eq. (1) averaged over the distribution  $\Phi(p)$ ), the correction to first order in  $1/m_b$  is eliminated from the total width,

$$\Gamma_{ACM} = \frac{\lambda^2 (m_b^{ACM})^3}{4\pi} \left( 1 + \mathcal{O}(1/m_b^2) \right), \quad (19)$$

in full agreement with the general statement of the absence of the  $1/m_b$  correction in the total width [7, 1, 8]. Thus, one must identify

$$M_B - m_b = \bar{\Lambda} = \frac{p_F\rho}{\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right) \rightarrow \begin{cases} 2p_F/\sqrt{\pi} & \text{at } \rho \rightarrow 0 \\ m_{sp} & \text{at } \rho \rightarrow \infty \end{cases}. \quad (20)$$

That the first order correction to the total width can be absorbed into model parameters is not surprising. What is less obvious is that this definition is compatible with QCD predictions for other quantities as well. We will show now that this is indeed the case. (The same conclusion concerning inclusive semi-leptonic decays has been reached previously in Refs. [9, 10].)

From the QCD analysis of Refs. [2, 11] we know the average photon energy that points to the true mass  $m_b$  :

$$\langle \epsilon \rangle = \langle M_B - 2E \rangle = \bar{\Lambda}(1 - \langle x \rangle) = \bar{\Lambda} \quad (21)$$

where  $x$  is defined by eq. (6) and  $\langle x \rangle = a_1 = 0$ . To find  $\langle \epsilon \rangle$  in the  $AC^2M^2$  model we use eq. (12) or eq. (15). Now the difference between the quark mass and  $M_B$  can be ignored and one obtains

$$\langle \epsilon \rangle_{ACM} = \frac{p_F\rho}{\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right) (1 + \mathcal{O}(1/m_b)). \quad (22)$$

which agrees with the QCD expression eq. (21) provided that the identification of eq. (20) is made. Moreover, this holds true for any choice of  $\Phi(p)$ .

For the second moment of the photon energy one derives from eq. (12)

$$\langle \epsilon^2 \rangle_{ACM} = p_F^2(2 + \rho). \quad (23)$$

On the other hand, according to Ref. [2]

$$\langle \epsilon^2 \rangle = \bar{\Lambda}^2(1 + \langle x^2 \rangle), \quad (24)$$

where  $\langle x^2 \rangle$  is related to the average kinetic energy of the  $b$  quark inside the  $B$  meson, see eqs. (56) and (94) in [2],

$$\langle x^2 \rangle = \frac{1}{3\bar{\Lambda}^2}(2M_B)^{-1}\langle B|\bar{b}\vec{\pi}^2 b|B\rangle \equiv \frac{\mu_\pi^2}{3\bar{\Lambda}^2}. \quad (25)$$

Comparing eqs. (23) and (24) we conclude that

$$p_F^2(2 + \rho) = \bar{\Lambda}^2(1 + \langle x^2 \rangle) = \frac{p_F^2 \rho^2}{\pi} e^\rho K_1^2(\frac{\rho}{2})(1 + \langle x^2 \rangle). \quad (26)$$

Assuming that  $\langle x^2 \rangle$  is known from a QCD analysis – and to a certain extent this is indeed the case – we can use eq. (26) to fix the value of  $\rho$  which then may serve as an input in eq. (20) allowing one to determine  $p_F$ . Moreover, rewriting eq. (26) in the form

$$1 + \langle x^2 \rangle = \pi(2 + \rho)\rho^{-2}e^{-\rho}K_1^{-2}(\frac{\rho}{2}) \quad (27)$$

it is easy to see that the right-hand side is a monotonously decreasing function of  $\rho$  having its maximum ( $\approx 1.6$ ) at  $\rho = 0$ . Thus, in the AC<sup>2</sup>M<sup>2</sup> model an upper bound emerges,

$$\langle x^2 \rangle < \frac{\pi}{2} - 1 \simeq 0.57 \quad (28)$$

On the other hand, the value of  $\mu_\pi^2$  has been estimated [12] from QCD sum rules [13] to be  $\mu_\pi^2 \sim 0.6$  GeV<sup>2</sup> and  $\bar{\Lambda} \simeq 0.4 \div 0.6$  GeV. Taking these estimates at face value one would conclude that  $\langle x^2 \rangle \sim 0.5 \div 1$ . This would mean that the AC<sup>2</sup>M<sup>2</sup> ansatz can be made consistent with the real QCD description only for small values of  $\rho$ , i.e. when the spectator is relativistic. Further numerical estimates will be presented below.

The higher moments are obtained in a straightforward way,

$$\langle \epsilon^n \rangle_{ACM} = \frac{p_F^n}{\sqrt{\pi}} \rho^{\frac{n+1}{2}} e^{\frac{\rho}{2}} K_{\frac{n+1}{2}}(\frac{\rho}{2}). \quad (29)$$

For even values of  $n$  the McDonald function in the right-hand side reduces to an elementary one,

$$\langle \epsilon^{2k} \rangle = p_F^{2k} \sum_{l=0}^k \frac{(2k-l)!}{l!(k-l)!} \rho^l. \quad (30)$$

For any value of  $n$  the limits of small and large  $\rho$  are

$$\langle \epsilon^n \rangle \rightarrow \begin{cases} 2^n p_F^n \Gamma(\frac{n+1}{2}) / \sqrt{\pi} & \text{at } \rho \rightarrow 0 \\ p_F^n \rho^{n/2} (1 + \frac{n(n+2)}{4\rho}) = m_{sp}^n (1 + \frac{n(n+2)}{4} \frac{p_F^2}{m_{sp}^2}) & \text{at } \rho \rightarrow \infty \end{cases}. \quad (31)$$

The moments  $\langle \epsilon^n \rangle$  given in eq. (29) are related to those of an ‘equivalent’ AC<sup>2</sup>M<sup>2</sup> distribution function *via* the following expression

$$\langle \epsilon^n \rangle = \bar{\Lambda}^n \langle (1-x)^n \rangle . \quad (32)$$

In particular combining eq. (29) for  $n = 3$  and eq. (27) we can express the third moment,  $\langle -x^3 \rangle$ , via another monotonously decreasing function of  $\rho$  implying that

$$0 < -\langle x^3 \rangle < 2 - \frac{\pi}{2} \simeq 0.43 ; \quad (33)$$

for small  $\rho$  the upper bounds of eqs. (28), (33) actually become approximate equalities. One should note that the QCD estimate [2] (see also [14]),

$$\langle -x^3 \rangle \approx 0.03 \text{ GeV}^3 / \bar{\Lambda}^3 \sim 0.3 \div 0.5 ,$$

is then quite compatible with the AC<sup>2</sup>M<sup>2</sup> value.

4. The equivalent ‘light cone’ distribution function  $F(x)$  – the Roman function – can be read off from eq. (15) taking into account the definition of the scaling variable  $\epsilon = \bar{\Lambda}(1-x)$ ,

$$F_{Rom}(x) = \frac{1}{\sqrt{\pi}} \frac{\bar{\Lambda}}{p_F} \exp \left\{ -\frac{1}{4} \left[ \frac{p_F}{\bar{\Lambda}} \frac{\rho}{1-x} - \frac{\bar{\Lambda}}{p_F} (1-x) \right]^2 \right\} , \quad (34)$$

$$\frac{\bar{\Lambda}}{p_F} = \frac{\rho}{\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right) .$$

This expression represents how the AC<sup>2</sup>M<sup>2</sup> ansatz can be reproduced by a QCD light cone function. The distribution given by the Roman function of eq. (34) satisfies all requirements necessary for the generic distribution function governing inclusive decays into massless quarks. As was mentioned, it does not lead to any spectrum beyond the kinematical boundary  $M_B/2$  and it exponentially decreases towards large negative  $x$ . Of course the latter property for  $F(x)$  must be considered as a goal in the mature QCD description incorporating radiative corrections rather than an obvious property; on the other hand it is most natural to require it from the model function introduced to describe non-perturbative effects.

If  $m_{sp} \gg p_F$  the Roman function  $F_{Rom}$  exhibits a narrow peak around  $x \simeq 0$  and represents the situation when even the spectator is nonrelativistic. The opposite limiting case is more relevant,  $p_F \gtrsim m_{sp}$ . In this case  $F_{Rom}$  is rather broad – its width is  $\sim \bar{\Lambda}$  – and very asymmetric, see Fig. 1.

5. Next we extend our analysis to inclusive semi-leptonic  $B$  decays. The question which we want to address here is supplementary to the one treated in Refs. [9, 10]. Namely, we are interested in studying the scaling features of the differential distribution

$$d\Gamma(B \rightarrow l\bar{\nu}_l X_u) / dE_l dq^2 dq$$

where  $E_l$  is the charged lepton energy and  $q_\mu = (p_l + p_\nu)_\mu$  is the total four-momentum of the lepton pair. The analysis of Ref. [2] implies that

$$m_b^5 \frac{d\Gamma(B \rightarrow l\bar{\nu}_l X_u)}{dE_l dq^2 dq_0} = \Gamma_0(b \rightarrow l\bar{\nu}_l u) \frac{2}{\Lambda} F(x) \frac{12(q_0 - E_l)(2m_b E_l - q^2)}{(m_b - q_0)} \quad (35)$$

where

$$\Gamma_0(b \rightarrow l\bar{\nu}_l u) = |V_{ub}|^2 \frac{G_F^2 m_b^5}{192\pi^3}. \quad (36)$$

In other words, we predict that the differential distribution – which generically is a function of two independent variables,  $q_0$  and  $q^2$  – actually depends only on a single scaling combination,

$$x = -\frac{m_b^2 + q^2 - 2m_b q_0}{2\bar{\Lambda}(m_b - q_0)} \quad (37)$$

apart from a simple kinematical factor. Like in deep inelastic scattering, this scaling holds only when both perturbative and higher twist effects are neglected. Both effects introduce corrections to the scaling regime which are not discussed here.

We would like to check whether this scaling feature – the dependence on a single scaling variable – persists in the AC<sup>2</sup>M<sup>2</sup> model. To this end we just repeat the steps carried out above for the radiative transition, see eq. (12). After averaging over the direction of  $\vec{p}$ , the primordial momentum of the  $b$  quark, we get for  $d^3\Gamma/dE_l dq^2 dq_0$  an integral representation quite similar to eq. (12). What is different is the lower limit  $p_0^2$  of integration over  $p^2$  which now takes the form

$$p_0^2 = \frac{1}{4} \left( \frac{m_{sp}^2}{\tilde{\epsilon}} - \tilde{\epsilon} \right)^2, \quad (38)$$

$$\tilde{\epsilon} = M_B \frac{M_B^2 + q^2 - 2q_0 M_B}{M_B^2 - q^2}. \quad (39)$$

This expression is sufficient to see that the dependence on  $q^2$  and  $q_0$  enters only via the variable  $\tilde{\epsilon}$ .

Moreover,  $\tilde{\epsilon}$  is directly expressed in terms of the QCD variable  $x$  of eq. (37),

$$\tilde{\epsilon} = \bar{\Lambda}(1-x) \left[ 1 + \mathcal{O}\left(\frac{\bar{\Lambda}}{m_b}(1-x)\right) \right]. \quad (40)$$

The AC<sup>2</sup>M<sup>2</sup> model, thus, reproduces the correct QCD scaling taking place in the leading approximation.

6. There is, however, an important feature in the true QCD description that is beyond the scope of the AC<sup>2</sup>M<sup>2</sup> ansatz: this model, as it is formulated in [5], is supposed to be universally applicable to the case of massless and of massive quarks in the final state, i.e. to  $b \rightarrow u$  and to  $b \rightarrow c$  transitions, respectively. On the other hand, we have shown [2] that the genuine distribution function depends in an essential way on the ratio of the quark masses in the initial and the final state,

$$F = F(x, \gamma) , \quad \gamma = m_q^2/m_b^2.$$

The function is strikingly different in the two extreme cases when  $\gamma \ll 1$  and  $1-\gamma \ll 1$ ; the latter represents the so-called small velocity (SV) limit [15]. Let us demonstrate that in this limit the AC<sup>2</sup>M<sup>2</sup> model would lead to predictions which differ essentially from the true QCD result.

In the SV limit the velocity of the final quark  $q$  is small,  $|\vec{v}| \ll 1$ ; this can happen both in radiative transitions and in semi-leptonic decays; in the latter case it requires

$$0 < (m_Q - m_q)^2 - q^2 \ll m_q m_Q .$$

and in the former

$$\Delta m = m_Q - m_q \ll m_Q .$$

When one retains only terms  $\mathcal{O}(v^0)$ , the physical spectrum in the  $B$  meson decay is exactly the same as it was at the free quark level – the  $\delta$ -function peak resides at the same place, and the inclusive probability is completely saturated by one heavy meson in the final state [15]. Notice that there is no ‘ $M_B - m_b$  window’ – no shift is present between the maximal allowed energies of the photon (or lepton) at the quark and the hadron levels.

Modifications of this perfect quark-hadron duality start at the level of  $\mathcal{O}(v^2)$  [16]. If  $\mathcal{O}(v^2)$  effects are considered the height of the elastic peak is changed, and a comb of inelastic peaks appears, the height of the latter being proportional to  $v^2$ . This comb will lie at  $E < E_0$  and will be stretched over an energy interval of order  $\bar{\Lambda}$ . ( $\bar{\Lambda}$  is not a very relevant parameter in this limit; we can continue to use it, however, just as a typical hadronic scale. One could certainly choose another definition of the typical hadronic scale.) The integral over the inelastic peaks must compensate the distortion of the elastic one – the so-called Bjorken sum rule [16]. In the recent paper [2] we demonstrated how this apparently different description emerges from the same QCD analysis that leads to the AC<sup>2</sup>M<sup>2</sup> -like prediction for massless final-state quarks.

Let us compare this picture to the one from the AC<sup>2</sup>M<sup>2</sup> model. Doppler smearing yields again one smooth peak whose width is now of order  $\bar{\Lambda}v$ . The AC<sup>2</sup>M<sup>2</sup> prescription thus produces nothing that resembles the two-component picture outlined above. The failure of this description has a clear origin – it incorporates Fermi motion in the initial state but disregards it in the final state where it is now crucial.

For the same reason in the actual applications of the model in  $b \rightarrow c$  transitions one has to introduce an *ad hoc* upper cut off for the heavy quark momentum, as was mentioned in the Introduction. This cut off is determined by the physical kinematic boundary and is, thus, non-universal, i.e. depends on the final quark mass.

The energy spectrum in the SV limit of QCD is given by a ‘temporal’ distribution function having in general a discrete support representing the higher excited states and supplemented by continuum contributions with at least two extra pions in the final state. One then has an Isgur-Wise -like [17] description of the spectrum where instead of relying on the QCD expressions for inclusive widths one is suggested to sum explicitly over the particular final state hadrons. To leading order in  $|\vec{v}|^2$  one then has only  $D^*(D)$  and in the next order adds higher excitations <sup>2</sup>. Adopting this description one then can fit the spectrum by a different phenomenological function not

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<sup>2</sup>This is similar to a modification of the original I-W ansatz, incorporating  $D^{**}$  states, as now used in experimental analyses of semileptonic spectra.

directly related to the Fermi motion; it is important however that in the framework of the QCD expansion there are certain relations between the moments of the Fermi motion distribution function and the parameters of the temporal one (cf. Ref. [2]). These relations must be observed in comparing decay spectra with massless and with heavy quarks in the final state.

The mass of the actual  $c$  quark is such that one might think at first sight it could be safely treated as massless since  $m_c^2/m_b^2 \sim 0.1$ . This naive expectation seems to be erroneous, though. For the true parameter that enters in semi-leptonic decays is rather given by the ratio

$$\tilde{\gamma} = \frac{4m_b^2 m_c^2}{(m_b^2 + m_c^2 - q^2)^2}$$

and its value is typically not very small even in the upper end of the charged lepton energy once it is integrated over the invariant mass squared of the lepton pair  $q^2$ . The conjecture that the  $c$  quark actually lies rather close to the SV limit is supported by the observation that the hadronic final state in  $B$  meson semi-leptonic decays consists, to large degree, of only a  $D$  or a  $D^*$ . (In the exact limit, semileptonic  $B$  decays would produce only a  $D$  or a  $D^*$  [15].) If so, it is clear that the attempts to fit the parameters of the AC<sup>2</sup>M<sup>2</sup> model directly from the data on the inclusive  $b \rightarrow c$  transitions, a quite popular procedure, are not very meaningful from a truly theoretical perspective. One can further get convinced that this is indeed the case by estimating the value of  $m_c$  as it emerges from these fits – it turns out to be too high from any standpoint.

It has been claimed in Ref. [3] that the same description in terms of shape functions holds for both  $b \rightarrow u$  and  $b \rightarrow c$  semileptonic transitions, and that the same hadronic matrix elements define it for an arbitrary ratio  $m_c^2/m_b^2$ . It is clear from the analysis of Ref. [2] that such claims are erroneous.

7. The intrinsic inadequacy of the AC<sup>2</sup>M<sup>2</sup> ansatz for describing semileptonic  $b \rightarrow c$  transitions in the SV limit gets obscured once one integrates over the neutrino momentum to obtain the charged lepton energy spectrum. Fitting  $B \rightarrow l + X_c$  by the AC<sup>2</sup>M<sup>2</sup> model one has commonly set  $m_{sp}$  to 150 MeV in a more or less ad-hoc fashion; the charm mass together with  $p_F$  are then used as free fit parameters. In an attempt to give this model a somewhat closer connection to QCD one should actually adopt a different strategy: namely to require that the difference between the AC<sup>2</sup>M<sup>2</sup> average of the  $b$  quark mass and the charm quark mass satisfy the same relationship with heavy flavor hadron masses that the heavy quark expansion yields in QCD for  $m_b - m_c$ , i.e.

$$\langle m_b \rangle_{ACM} - m_c \simeq \frac{3M_B^* + M_B}{4} - \frac{3M_D^* + M_D}{4} + \langle \bar{\pi}^2 \rangle \frac{m_b - m_c}{2m_b m_c} \quad (41)$$

Instead one allows  $m_{sp}$  together with  $p_F$  to float in the fit. It is quite conceivable that such an approach would yield values for the fit parameters in better agreement with QCD expectations. We do not have at hand the results of such an analysis; still, to have an example of the possible distribution function one can use the existing fit parameters from CLEO data [18], namely  $p_F \simeq 282$  MeV and  $m_{sp} \simeq 150$  MeV. From

e.qs. (20-21, 29, 32) we get

$$\begin{aligned}\bar{\Lambda} &\simeq 360 \text{ MeV} , \quad \langle \vec{\pi}^2 \rangle \simeq 0.16 \text{ GeV}^2 \\ a_2 &\simeq 0.424 , \quad a_3 \simeq -0.28 , \quad a_4 \simeq 0.71 , \quad a_5 \simeq -1.2\end{aligned}\tag{42}$$

and the distribution function itself in terms of the photon energy is shown in Fig. 2. Keep in mind that this curve should be taken only as an illustration since the perturbative gluon corrections, very important for the numerical analysis, are not included here.

One should note that the value of the kinetic energy operator is slightly below the lower bound derived in Ref. [2]. This points again to the intrinsic inconsistency of the literal  $AC^2M^2$  approach with semileptonic decays into charm. The third moment of the distribution functions is somewhat less than the estimate of Ref. [2] as well. All these facts seem to be correlated with the too high mass of the  $c$  quark obtained in the  $AC^2M^2$  fit of such decays. In reality one can obtain better information on the distribution function  $F(x)$  at small  $\gamma$  studying the double differential distributions and in particular in the region of small  $q^2$  [19, 2]. It is worth reminding however that according to the analysis of Ref. [1] the calculable  $1/m_Q^2$  corrections are really important for the fit and they must be taken into account.

8. So far we have disregarded gluon radiative corrections that actually modify the end point shape in an essential way; their interplay with the effects of the Fermi motion has been discussed in ref. [2]. In radiative decays of the type  $b \rightarrow s + \gamma$  the peak in the photon energy is still manifest although its height is significantly lowered by gluon bremsstrahlung. Both perturbative and non-perturbative effects smearing the original narrow line can be described with sufficient practical accuracy by convoluting the radiatively corrected parton spectrum with the distribution function describing Fermi motion [2]; the radiative corrections must be considered in higher orders of perturbative expansion as well, at least in the double logarithmic approximation. A more precise treatment including subleading perturbative corrections requires further theoretical consideration. We note that it is just this theoretical prescription that has been employed in the existing phenomenological analysis (for radiative decays, see Ref. [20]). In semileptonic decays integrating over the neutrino energy smooths out the spectrum and the impact of radiative corrections on the shape is less pronounced.

The conjecture in Refs. [3, 4] that radiative corrections produce relatively small, namely  $\mathcal{O}(\alpha_s(m_b))$ , modifications to the moments of the end-point distribution functions is certainly incorrect.

9. In the conclusion of this paper we would like to add some comments on existing comparisons of the predictions from the  $AC^2M^2$  model with QCD. The most obvious remark follows from the fact that there are no corrections of order  $1/m_b$  to total widths, as stated first in Refs. [7, 8]; thus, for example, differences in the lifetimes of  $B$  and  $\Lambda_b$  arise first on the  $1/m_b^2$  level; the corrections due to Fermi motion are quadratic in  $1/m_b$  as well. This does not hold automatically in  $AC^2M^2$ ; yet by defining an effective  $b$  quark mass  $m_b^{ACM}$  as the average over the floating mass of eq. (1) one eliminates  $1/m_b$  corrections to both the total width and to the regular part of the

lepton spectrum. This observation has been noted in a few recent papers [9, 10]. The above analysis shows that the similarity extends also to the subtle problem of the endpoint description for decays into light quarks – contrary to superficial claims in Refs. [3, 4]. As a matter of fact the corresponding considerations of Refs. [3, 4] do not refer to the AC<sup>2</sup>M<sup>2</sup> model at all since the crucial effect of the floating mass  $m_b^f$  has been omitted. As a result the kinematical boundaries were violated and the numerical values of the moments obtained there are irrelevant. In contrast, the analysis of  $b \rightarrow s + \gamma$  in Ref. [20] has been done properly: e.g. it preserved the kinematic constraints.

Of course essential differences arise at order  $1/m_Q^2$ ; however one should keep in mind that the main point of the AC<sup>2</sup>M<sup>2</sup> model was to take care of only leading  $1/m_Q$  effects that are crucial even in beauty particles.

In our opinion, this is actually not the main difference, probably not even from a practical point of view. For it is important to understand that the value of the  $b$  quark mass  $m_b^{ACM}$  defined in such a way can actually appear to be *different* for different beauty hadrons  $H_b$  and even for the different channels ( $b \rightarrow c$  vs.  $b \rightarrow u$  or  $b \rightarrow s + \gamma$ ) in the same hadron. The latter concern follows from the fact that the spectrum is shaped by different distribution functions depending on the final state quark mass; only the second moments of these distribution functions coincide. This difference can actually lead to the emergence of different values for  $m_b^{ACM}$  beyond the nonrelativistic approximation  $p_F \ll m_{sp}$ .

Once semileptonic or radiative decays of beauty baryons are studied, the second concern arises: it is more than conceivable that the literal use of the AC<sup>2</sup>M<sup>2</sup> fit for the decays of baryons and mesons would yield values of  $m_b^{ACM}$  that differ already in  $1/m_Q$  terms; following the standard AC<sup>2</sup>M<sup>2</sup> rules for calculating the width one then would arrive at different lifetimes already in the leading non-perturbative approximation. For the average mass of the heavy quark plays only a marginal role in the model, and thus essentially depends on the assumed shape of the distribution. The whole distribution function  $F(x)$  can be determined from a precise measurement of the end point spectrum, in particular in  $b \rightarrow s + \gamma$  decays. This of course would yield the true  $b$  quark mass. On the other hand lepton spectra in semi-leptonic decays are less sensitive to the exact shape and thus effectively rely more on the model assumptions. For in general there is no reason to expect that the particular AC<sup>2</sup>M<sup>2</sup> ansatz for the distribution function holds with a good accuracy for real boundstates in the presence of potential-like binding forces. For example, a possible and very natural generalization of the ansatz eq. (1) would be to add some arbitrary “static” binding energy  $V$  of the order of  $\Lambda_{QCD}$ :

$$M_B \simeq m_b^f + \sqrt{|\vec{p}|^2 + m_{sp}^2} + V \quad (43)$$

where  $V$  may differ from hadron to hadron. This modification cannot be absorbed into a redefinition of  $m_{sp}$  and  $p_F$ . No model can actually provide us with an unambiguous prescription for determining the impact of  $V$ ; QCD on the other hand ensures that the effect of  $V$  on the total width is cancelled by an analogous interaction in the final state. Therefore it seems to be advantageous to impose additional constraint on the fitted distribution functions ensuring them to yield the same QCD value of the

‘average’ mass  $m_b$  independently of a particular ansatz used for their shape.

We think that the possibility to obtain different mass values for the same heavy quark is actually the main *physical* indication of the inconsistency of the standard  $AC^2M^2$  ansatz with QCD, together with a difference in the underlying Fermi motion description for different final state quark masses – which in the case of the SV limit is in a sense a rather obvious observation.

**ACKNOWLEDGEMENTS:** We thank G. Altarelli for his thoughtful comments on the manuscript. N.U. acknowledges interesting discussions with T. Mannel and is grateful to D. Cassel for his interest. M.S. and A.V. are grateful to M. Voloshin for useful comments. We are also indebted to R. Mountain for producing the figures. This work was supported in part by the National Science Foundation under the grant number PHY 92-13313 and by DOE under the grant number DOE-AC02-83ER40105.

## References

- [1] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, *Phys. Rev. Lett.* **71** (1993) 496.
- [2] I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, **preprint** CERN-TH.7129/93, 1993.
- [3] M. Neubert , **preprint** CERN-TH.7087/93, 1993.
- [4] M. Neubert , **preprint** CERN-TH.7113/93, 1993.
- [5] G. Altarelli et al., *Nucl. Phys.* B208 (1982) 365.
- [6] J. Chay, H. Georgi and B. Grinstein, *Phys. Lett.* **B247** (1990) 399.
- [7] I. Bigi, N. Uraltsev and A. Vainshtein, *Phys. Lett.* **B293** (1992) 430; (E) B297 (1993) 477;  
B. Blok and M. Shifman, *Nucl. Phys.* **B399** (1993) 441; 459;
- [8] I. Bigi, B. Blok, M. Shifman, N. Uraltsev and A. Vainshtein, *Proc. of the 1992 DPF meeting of APS*, Fermilab, November 1992 [Preprint UND-HEP-92-BIG07].
- [9] C. Csaki and L. Randall, **preprint** CTP-2262, 1993.
- [10] G. Baillie, **preprint** UCLA/93/TEP/47.
- [11] A. Falk, M.Luke, M.Savage, **preprint** UCSD/PTH 93-23.
- [12] P. Ball and V. Braun, **Preprint** MPI-Ph/93-51, 1993.
- [13] For a review see M. Shifman, Ed., *Vacuum Structure and QCD Sum Rules*, North-Holland, 1992.
- [14] T.Mannel, private communication; *in preparation*.

- [15] M. Voloshin and M. Shifman, *Yad. Fiz.* **47** (1988) 801 [*Sov. J. Nucl. Phys.* **47** (1988) 511].
- [16] J. Bjorken, Invited Talk at *Les Rencontres de la Valle d'Aosta, La Thuille, 1990* Preprint SLAC-PUB-5278, 1990;  
see also N. Isgur and M. Wise, *Phys. Rev.* **D43** (1991) 819.
- [17] B. Grinstein, N. Isgur, M.B. Wise, *Phys. Rev. Lett.* **56** (1986) 258.
- [18] J. Bartelt et al. (CLEO collab.) “Inclusive Measurement of B Meson Semileptonic Branching Fractions”, Cornell Univ. Preprint CLEO CONF 93-19. 1993.
- [19] B. Blok, L. Koprakh, M. Shifman and A. Vainshtein, Preprint TPI- MINN-93/33-T [Phys. Rev. D, to be published];  
A. Manohar and M. Wise, Preprint UCSD/PTH 93-14;  
T. Mannel, Preprint IKDA 93/26.
- [20] A. Ali, C. Greub, *Zeit. Phys.* **C49** (1991) 431; *Phys. Lett.* **B259** (1991) 182; for recent update, *Zeit. Phys.* **C60** (1993) 433.

### Figure captions

Fig. 1. The QCD distribution function  $F(x)$  from the AC<sup>2</sup>M<sup>2</sup> model (Roman function) for  $\rho = 1$  (solid line),  $\rho = 0.2$  (dotted line) and  $\rho = 5$  (dashed line);  $\rho = m_{sp}^2/p_F^2$ .

Fig. 2. The distribution over the light cone momentum  $\pi_0 + \pi_z$  corresponding to the AC<sup>2</sup>M<sup>2</sup> model with  $p_F = 282$  MeV and  $m_{sp} = 150$  MeV.

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